

FROBENIUS SPLITTING AND MÖBIUS INVERSION

ALLEN KNUTSON

ABSTRACT. We show that the fundamental class in K-homology of a Frobenius split scheme can be computed as a certain alternating sum over irreducible varieties, with the coefficients computed using Möbius inversion on a certain poset.

If G/P is a generalized flag manifold and X is an irreducible subvariety homologous to a multiplicity-free union of Schubert varieties, then using a result of Brion we show how to compute the K_0 -class $[X] \in K_0(G/P)$ from the Chow class in $A_*(G/P)$.

CONTENTS

1. Statement of results	1
Acknowledgements	2
2. Proofs	3
References	5

1. STATEMENT OF RESULTS

Let X be a Noetherian scheme, and let \mathcal{P} be a finite set of (irreducible) subvarieties of X , with the following **intersect-decompose** property: for any subset $S \subseteq \mathcal{P}$, the geometric components of $\bigcap S$ should also be elements of \mathcal{P} . (In particular, if $S = \emptyset$ we interpret $\bigcap S$ as X , and require that \mathcal{P} contain X 's geometric components.) Let \mathcal{P}_X denote the obvious minimal such \mathcal{P} , constructed from X 's geometric components by intersecting and decomposing until done.

For example, let $X = \{(x, y, z) : y(yz^2 - x^2(x - z)) = 0\}$. This has two components, $A := \{y = 0\}$ and $B := \{yz^2 = x^2(x - z)\}$. Their (nonreduced) intersection is $\{y = x^2(x - z) = 0\}$, which has geometric components $C := \{y = x = 0\}$ and $D := \{y = x - z = 0\}$. Finally, $C \cap D = \{\vec{0}\}$. So $\mathcal{P}_X = \{A, B, C, D, \{\vec{0}\}\}$.

Note that in this example, even though X was reduced (and even Cohen-Macaulay) one ran into nonreducedness when one started intersecting components. There is a well-known condition that allows one to avoid this:

Lemma 1. *Let X be Frobenius split (for which our reference is [BrK05]). Then for any $A, \{B_i\} \in \mathcal{P}_X$, $A \cap \bigcup_i B_i$ is reduced.*

Proof. This is immediate from [BrK05, proposition 1.2.1]. □

Date: February 11, 2009.

Supported by an NSF grant.

The **Möbius function** of a finite poset P is the unique function $\mu_P : P \rightarrow \mathbb{Z}$ such that $\forall p \in P, \sum_{q \geq p} \mu(q) = 1$.

Theorem 1. *Let X be a reduced scheme such that for any $A, \{B_i\} \in \mathcal{P}_X$, $A \cap \bigcup_i B_i$ is reduced. Let $\mathcal{P} \supseteq \mathcal{P}_X$ be a collection of subvarieties with the intersect-decompose property. For each $A \in \mathcal{P}$, let $[A] \in K_0(X)$ denote the K -homology class of the structure sheaf of A . Then*

$$[X] = \sum_{A \in \mathcal{P}} \mu_{\mathcal{P}}(A) [A].$$

(In fact $\mu_{\mathcal{P}}(A) = 0$ unless $A \in \mathcal{P}_X$, in which case $\mu_{\mathcal{P}}(A) = \mu_{\mathcal{P}_X}(A)$.)

Assume now that X carries an action of a group G . Assume too that G preserves each element of \mathcal{P} ; this is automatic if $\mathcal{P} = \mathcal{P}_X$ and G is connected. Then the classes in G -equivariant K -homology obey exactly the same formula above.

It probably appears superfluous at this point to allow \mathcal{P} to be any larger than \mathcal{P}_X , insofar as it doesn't change the formula above. The recursive definition of \mathcal{P}_X makes it difficult to compute, however, and sometimes it is easier to give an upper bound. For example, if Y is a scheme carrying an action of a group B with finitely many orbits, and $X \subseteq Y$ is closed and B -invariant, then we can take \mathcal{P} to be the set of B -orbit closures contained in X .

In [Br03] was proven the following remarkable fact:

Theorem 2. *Let X be a subvariety (i.e. reduced and irreducible subscheme) of a generalized flag manifold G/P . Assume that the Chow class $[X]_{\text{Chow}} \in A(G/P)$ is a sum of Schubert classes $\sum_{d \in D} [X_d]_{\text{Chow}}$, with no multiplicities. (Here D is a subset of the Bruhat order W/W_P .)*

Then there is a flat degeneration of X to the reduced union $\bigcup_{d \in D} X_d$, and both subschemes are Cohen-Macaulay.

Combining this with the theorem above, we will obtain

Theorem 3. *Let X be a multiplicity-free subvariety of G/P , in the sense of [Br03], with $[X]_{\text{Chow}} = \sum_{d \in D} [X_d]_{\text{Chow}}$. Let $\mathcal{P} \subseteq W/W_P$ be the set of Schubert varieties contained in $\bigcup_{d \in D} X_d$ (an order ideal in the Bruhat order on W/W_P). Then as an element of $K_0(G/P)$,*

$$[X] = \sum_{X_e \subseteq \bigcup_{d \in D} X_d} \mu_{\mathcal{P}}(X_e) [X_e].$$

Note that the X in the last theorem above is *not* assumed to be Frobenius split. (Its degeneration $\bigcup_{d \in D} X_d$ is, automatically [BrK05, theorem 2.2.5].)

The preprint [Sn] applies our theorem 3 to the case that X is a multiplicity-free Richardson variety in a Grassmannian, giving an independent proof of Buch's K -theoretic Littlewood-Richardson rule [Bu02] in the case that the ordinary product is multiplicity-free.

In [KLS] we will use theorem 1 to compute the K -classes of the closed strata in the cyclic Bruhat decomposition, whose study was initiated in [Po] and continued in e.g. [Wi05, PSW, LW08].

Acknowledgements. We thank Michelle Snider for many useful conversations, and most especially for the insight that the second half of lemma 2 should be traced to the first.

2. PROOFS

We first settle the difference between the poset \mathcal{P}_X and more general posets \mathcal{P} , with a combinatorial lemma we learned from Michelle Snider.

Lemma 2. *Let $P \supseteq Q$ be two finite posets such that $\forall S \subseteq Q$, all the greatest lower bounds in P of S are also in Q . (In particular, the $S = \emptyset$ case implies that Q contains all of P 's maximal elements.) Then*

- (1) *for each $p \in P \setminus Q$, the set $Q_p = \{q \in Q : q \geq p\}$ has a unique minimal element, and*
- (2) *$\mu_P(p) = \mu_Q(p)$ for $p \in Q$, and otherwise $\mu_P(p) = 0$.*

Proof. (1) Let $p \notin Q$. Let $S = \{s \in P : \forall q \in Q_p, q \geq s\}$. Tautologically, Q_p is an upward order ideal, S is a downward order ideal, and $S \ni p$. By assumption, Q contains the maximal elements of S . Pick one that is larger than p and call it q_{\min} .

Since $q_{\min} \in S$, $q_{\min} \leq q'$ for all $q' \in Q_p$. Since $q_{\min} \geq p$ and $q_{\min} \in Q$, $q_{\min} \in Q_p$. So q_{\min} is the unique minimal element of Q_p .

- (2) Define $m : P \rightarrow \mathbb{Z}$ by $m(p) = \mu_Q(p)$ for $p \in Q$, and otherwise $m(p) = 0$. Our goal is to show that $\mu_P = m$, or equivalently, that m satisfies the defining criterion of Möbius functions: $\forall p \in P, \sum_{p' \in P, p' \geq p} m(p') = 1$.

Let $q_{\min} \geq p$ be the minimum element of Q_p . (It equals p iff $p \in Q$.) Then

$$\sum_{p' \in P, p' \geq p} m(p') = \sum_{p' \in Q, p' \geq p} m(p') = \sum_{p' \in Q_p} m(p') = \sum_{p' \in Q, p' \geq q_{\min}} m(p') = \sum_{p' \in Q, p' \geq q_{\min}} \mu_Q(p') = 1.$$

□

The following lemma establishes the property of Möbius functions that we will use to connect them to K -classes.

- Lemma 3.** (1) *Let P be a finite poset, and Q a downward order ideal. Extend μ_Q to P by defining $\mu_Q(p) = 0$ for $p \in P \setminus Q$. Then $\sum_{p' \geq p} \mu_Q(p') = [p \in Q]$, meaning 1 for $p \in Q$, 0 for $p \notin Q$.*
- (2) *Let P be a finite poset, with two downward order ideals P_1, P_2 such that $P = P_1 \cup P_2$. Extend $\mu_{P_1}, \mu_{P_2}, \mu_{P_1 \cap P_2}$ to functions on P by defining them as 0 on the new elements. Then $\mu_P = \mu_{P_1} + \mu_{P_2} - \mu_{P_1 \cap P_2}$.*

Proof. (1) For any $p \in P$,

$$\sum_{p' \in P, p' \geq p} \mu_Q(p') = \sum_{q \in Q, q \geq p} \mu_Q(q)$$

which is an empty sum unless $p \in Q$. If $p \in Q$, then it becomes the usual Möbius function sum for Q , so adds up to 1.

- (2) By the result above,

$$\sum_{q \geq p} (\mu_{P_1}(q) + \mu_{P_2}(q) - \mu_{P_1 \cap P_2}(q)) = [q \in P_1] + [q \in P_2] - [q \in P_1 \cap P_2].$$

If $q \in P_1 \setminus P_2$, this gives $1 + 0 - 0 = 1$; similarly if $q \in P_2 \setminus P_1$. If $q \in P_1 \cap P_2$, this gives $1 + 1 - 1 = 1$. These are all the cases, by the assumption $P = P_1 \cup P_2$.

Since

$$\sum_{q \geq p} (\mu_{P_1}(q) + \mu_{P_2}(q) - \mu_{P_1 \cap P_2}(q)) = 1$$

for all $p \in P$, this $\mu_{P_1} + \mu_{P_2} - \mu_{P_1 \cap P_2}$ must be the Möbius function μ_P .

□

Proof of theorem 1. First, we observe that $\mathcal{P}_X \subseteq \mathcal{P}$ satisfies the condition of lemma 2; for any collection S of varieties in \mathcal{P}_X , and $Y \in \mathcal{P}$ such that $Y \subseteq \bigcap S$, there exists $Y' \in \mathcal{P}_X$, $Y' \supseteq Y$. Proof: since Y is irreducible, it is contained in some geometric component Y' of $\bigcap S$, and by the recursive definition of \mathcal{P}_X we know $Y' \in \mathcal{P}_X$.

By part (2) of lemma 2,

$$\sum_{A \in \mathcal{P}} \mu_{\mathcal{P}}(A) [A] = \sum_{A \in \mathcal{P}_X} \mu_{\mathcal{P}_X}(A) [A].$$

So it suffices for the remainder to assume that $\mathcal{P} = \mathcal{P}_X$.

If X is irreducible, then $\mathcal{P}_X = \{X\}$, and the formula is easily verified:

$$\sum_{A \in \mathcal{P}_X} \mu_{\mathcal{P}_X}(A) [A] = \mu_{\mathcal{P}_X}(X) [X] = 1 [X] = [X].$$

This will be the base of an induction on the number of components; we assume hereafter that there are at least 2.

Let A be a geometric component of X , and X' the union of the other components. Then we have a formula on K-homology classes:

$$(1) \quad [X] = [A] + [X'] - [A \cap X'].$$

Let $P_1 = \{Y \in \mathcal{P}_X : Y \subseteq A\}$, $P_2 = \{Y \in \mathcal{P}_X : Y \subseteq X'\}$. Then by induction, the three terms on the right-hand side can be computed by Möbius inversion on $P_1, P_2, P_1 \cap P_2$.

Now apply part (2) of lemma 3 to say that

$$\mu_{\mathcal{P}_X} = \mu_{P_1} + \mu_{P_2} - \mu_{P_1 \cap P_2}.$$

Putting these together,

$$\begin{aligned} \sum_{C \in \mathcal{P}_X} \mu_{\mathcal{P}_X}(C) [C] &= \sum_{C \in \mathcal{P}_X} (\mu_{P_1}(C) + \mu_{P_2}(C) - \mu_{P_1 \cap P_2}(C)) [C] \\ &= \left(\sum_{C \in P_1} \mu_{P_1}(C) [C] \right) + \left(\sum_{C \in P_2} \mu_{P_2}(C) [C] \right) - \left(\sum_{C \in P_1 \cap P_2} \mu_{P_1 \cap P_2}(C) [C] \right) \\ &= [A] + [X'] - [A \cap X'] \\ &= [X]. \end{aligned}$$

If we intersect G-invariant subvarieties of X , the result is again G-invariant. If G is connected, hence irreducible, then it preserves each component of any G-invariant subvariety. Hence by induction G preserves each element of \mathcal{P}_X . G-equivariant K-homology also satisfies equation (1), and the remainder of the argument is the same. □

Proof of theorem 3. The K-class is preserved under flat degenerations, so $[X] = [\bigcup_{d \in D} X_d]$. By [BrK05, theorem 2.2.5], there is a Frobenius splitting on G/P for which $\bigcup_{d \in D} X_d$ is compatibly split. In particular, $\bigcup_{d \in D} X_d$ is Frobenius split, and lemma 1 applies.

To apply theorem 1, we need a collection \mathcal{P} of irreducible subvarieties of $\bigcup_{d \in D} X_d$, with the intersect-decompose property. So we take \mathcal{P} to be the set of Schubert varieties $\{X_e\}$ contained in $\bigcup_{d \in D} X_d$. Since the Schubert varieties are the orbit closures for the action of a Borel subgroup on G/P , any intersection $A \cap \bigcup_i B_i$ will again be Borel-invariant. Since that Borel acts with finitely many orbits, any Borel-invariant subvariety is an orbit closure. This shows that the components of any intersection $A \cap \bigcup_i B_i$ are in \mathcal{P} .

Now we apply theorem 1, and obtain the desired formula. \square

In the application in [Sn], the subvariety X is preserved under the action of the maximal torus T of G , and of course the Schubert varieties $\{X_d\}$ are as well. However, theorem 3 does *not* give an equality of T -equivariant K -homology classes, as the flat degeneration is not T -equivariant.

REFERENCES

- [Br03] M. Brion, Multiplicity-free subvarieties of flag varieties, Contemporary Math. 331, 13-23, Amer. Math. Soc., Providence, 2003. <http://arxiv.org/abs/math.AG/0211028>
- [BrK05] ———, S. Kumar, Frobenius Splitting Methods in Geometry and Representation Theory, Progress in Mathematics, 231. Birkhuser Boston, Inc., Boston, MA, 2005.
- [Bu02] A. Buch, A Littlewood-Richardson rule for the K -theory of Grassmannians, Acta Math. 189 (2002), no. 1, 37–78. <http://arxiv.org/abs/math.AG/0004137>
- [KLS] A. Knutson, T. Lam, D. Speyer, Positroid varieties I: juggling and geometry. In preparation.
- [LW08] T. Lam, L. Williams, Total positivity for cominuscule Grassmannians, New York Journal of Mathematics, Volume 14, 2008, 53–99. <http://arxiv.org/abs/arXiv:0710.2932>
- [Po] A. Postnikov, Total positivity, Grassmannians, and networks. <http://arxiv.org/abs/math.CO/0609764>
- [PSW] ———, D. Speyer, L. Williams, Matching polytopes, toric geometry, and the non-negative part of the Grassmannian. <http://arxiv.org/abs/arXiv:0706.2501>
- [Sn] M. Snider, A combinatorial approach to multiplicity-free Richardson subvarieties of the Grassmannian, preprint.
- [Wi05] L. Williams, Enumeration of totally positive Grassmann cells, Advances in Mathematics, Volume 190, Issue 2, January 2005, pages 319–342. <http://arxiv.org/abs/math.CO/0307271>

E-mail address: allenk@math.cornell.edu